Analytic Combinatorics Exercise Sheet 4

Exercises for the session on 17/5/2017

Problem 4.1

For $\delta > 0$, let $\mathcal{H}_\delta$ denote the anti-clockwise open contour in the complex plane comprised of the following three parts: the line $x+\delta i$ for $x \in \mathbb{R}^+$; the semi-circle $-\delta e^{i\theta}$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$; and the line $x - \delta i$ for $x \in \mathbb{R}^+$. Show

$$\lim_{\delta \to 0} \int_{\mathcal{H}_\delta} (-z)^{\frac{1}{2}} e^{-z} \, \mathrm{d}z = 2i \, \Gamma\left(3/2\right),$$

where the range of the principal argument should be taken as $(-\pi, \pi]$.

Problem 4.2

Let $G(z)$ denote the ordinary generating function for the class of unlabelled plane rooted trees. Use singularity analysis to find an asymptotic expression for $[z^n]G(z)$ (you may use the standard results

$$[z^n](1-z)^{-\alpha} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \tag{1}$$

and

$$\Gamma(-1/2) = -2\sqrt{\pi}. \tag{2}$$

in your working).

A general formula for trees with generating function $F(z) = z\phi(F(z))$ is

$$[z^n]F(z) \sim \frac{1}{\sqrt{\frac{2\pi\phi''(\lambda)}{\phi(\lambda)}}} (\phi'(\lambda))^n n^{-\frac{3}{2}}, \tag{3}$$

where λ is the positive real root of $\phi(u) = u\phi'(u)$. Evaluate the right-hand-side of (3) for $G(z)$, and check that this agrees with your answer.

Problem 4.3

Let $M(z)$ denote the ordinary generating function for the class of unlabelled plane rooted trees in which each vertex has at most two descendants. Use singularity analysis to find an asymptotic expression for $[z^n]M(z)$ (you may again use (1) and (2)), and then check your answer using (3).

Problem 4.4

Let $C(z)$ denote the exponential generating function for the class of labelled non-plane rooted trees. Use Lagrange's Inversion Theorem to determine an asymptotic expression for $n![z^n]C(z)$, and use this to prove Stirling's formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Problem 4.5

Let $R(z)$ denote the exponential generating function for the class of labelled 2-regular graphs. Use the exp-log schema to determine asymptotic expressions for $[z^n]R(z)$ and for the expected number of components in a random labelled 2-regular graph of size n.

Problem 4.6

A general formula for trees with generating function $F(z) = z\phi(F(z))$ is given by

$$F(z) \sim \lambda - \sqrt{\frac{2\phi(\lambda)}{\phi''(\lambda)}}\sqrt{1 - z\phi'(\lambda)}, \tag{4}$$

where λ is again the positive real root of $\phi(u) = u\phi'(u)$. Evaluate the right-hand-side of (4) for $C(z)$, where $C(z)$ is as defined in Problem 4.4.

Let $M(z)$ denote the exponential generating function for the class $\mathcal{M}$ of mappings from $\{1, 2, \ldots, n\}$ to itself, and use your expression for $C(z)$ to write $\mathcal{M}$ in exp-log form. Hence, use the exp-log schema to find an asymptotic expression for $n![z^n]M(z)$